INDEFINITE INTEGRATION [JEE ADVANCED PREVIOUS YEAR SOLVED PAPER]

JEE ADVANCED

Single Correct Answer Type

1. The value of the integral $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx$ is

a. $\sin x - 6\tan^{-1}(\sin x) + C$

b. $\sin x - 2(\sin x)^{-1} + C$

c. $\sin x - 2(\sin x)^{-1} - 6\tan^{-1}(\sin x) + C$

d. $\sin x - 2(\sin x)^{-1} + 5\tan^{-1}(\sin x) + C$

(IIT-JEE 1995)

2. $\int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}} dx$ is equal to

a. $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^3} + C$

b. $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x} + C$

c. $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^2} + C$

d. $\frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + C$ **(IIT-JEE 2006)**

3. Let $f(x) = \frac{x}{(1+x^n)^{1/n}}$ for $n \geq 2$ and $g(x) = \underbrace{(fofo\ldots of)}_{f \text{ occurs } n \text{ times}}(x)$.

Then $\int x^{n-2} g(x)\, dx$ equals

a. $\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}} + K$

b. $\frac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}} + K$

c. $\frac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}} + K$

d. $\frac{1}{n+1}(1+nx^n)^{1+\frac{1}{n}} + K$ **(IIT-JEE 2007)**

4. Let $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1} dx$, $J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1} dx$. Then for an arbitrary C, the value of $J - I$ equals

a. $\frac{1}{2}\log\left(\frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^{2x} + 1}\right) + C$

b. $\frac{1}{2}\log\left(\frac{e^{2x} + e^{x} + 1}{e^{2x} - e^{x} + 1}\right) + C$

c. $\frac{1}{2}\log\left(\frac{e^{2x} - e^{x} + 1}{e^{2x} + e^{x} + 1}\right) + C$

d. $\frac{1}{2}\log\left(\frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^{2x} + 1}\right) + C$ **(IIT-JEE 2008)**

5. The integral $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$ equals (for some arbitrary constant K)

a. $-\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

b. $\frac{1}{(\sec x + \tan x)^{1/11}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

c. $-\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

d. $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(IIT-JEE 2012)

Assertion-Reasoning Type

1. Let $F(x)$ be an indefinite integral of $\sin^2 x$.

Statement 1: The function $F(x)$ satisfies $F(x + \pi) = F(x)$ for all real x.

Statement 2: $\sin^2(x + \pi) = \sin^2 x$ for all real x.

a. Statement 1 is true, statement 2 is true; statement 2 is the correct explanation for statement 1.

b. Statement 1 is true, statement 2 is true; statement 2 is NOT the correct explanation for statement 1.

c. Statement 1 is true, statement 2 is false.

d. Statement 1 is false, statement 2 is true.

(IIT-JEE 2007)

Fill in the Blanks Type

1. $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Ax + B\log(9e^{2x} - 4) + C$, then

$A = $ ____, $B = $ ____, $C = $ ____. **(IIT-JEE 1990)**

Subjective Type

1. Evaluate $\int \frac{\sin x}{\sin x - \cos x} dx$. **(IIT-JEE 1978)**

2. Evaluate $\int \frac{x^2}{(a+bx)^2} dx$. **(IIT-JEE 1979)**

3. Evaluate the following integrals:

a. $\int \sqrt{1+\sin\left(\frac{x}{2}\right)}dx$ b. $\int \frac{x^2}{\sqrt{1-x}}dx.$

(IIT-JEE 1980)

4. Evaluate $\int (e^{\log x} + \sin x) \cos x \, dx.$ **(IIT-JEE 1981)**

5. Evaluate $\int \frac{(x-1)e^x}{(x+1)^3} dx.$ **(IIT-JEE 1983)**

6. Evaluate $\int \frac{dx}{x^2(x^4+1)^{3/4}}.$ **(IIT-JEE 1984)**

7. Evaluate $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx.$ **(IIT-JEE 1985)**

8. Evaluate $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx.$ **(IIT-JEE 1986)**

9. Evaluate $\int \frac{\sqrt{\cos 2x}}{\sin x} dx.$ **(IIT-JEE 1987)**

10. Evaluate $\int (\sqrt{\tan x} + \sqrt{\cot x})\, dx.$ **(IIT-JEE 1989)**

11. Evaluate $\int \left(\frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} + \frac{\ln\left(1+\sqrt[6]{x}\right)}{\sqrt[3]{x} + \sqrt{x}} \right) dx.$

(IIT-JEE 1992)

12. Evaluate $\int \cos 2\theta \ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) d\theta.$

(IIT-JEE 1994)

13. Evaluate $\int \frac{x+1}{x(1+xe^x)^2} dx.$ **(IIT-JEE 1996)**

14. Evaluate $\int \frac{1}{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx.$ **(IIT-JEE 1997)**

15. Evaluate $\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)} dx.$ **(IIT-JEE 1999)**

16. Evaluate $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right) dx.$ **(IIT-JEE 2001)**

17. Evaluate for, $m \in N$,

$\int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m} dx, x > 0.$

(IIT-JEE 2002)

Answer Key

JEE Advanced

Single Correct Answer Type

1. c. 2. d. 3. a. 4. c. 5. c.

Assertion–Reasoning Type

1. d

Fill in the Blanks Type

1. $A = -3/2, B = \frac{35}{36}$; C can have any real value.

Subjective Type

1. $\frac{1}{2}\log|\sin x - \cos x| + \frac{x}{2} + C$

2. $\frac{1}{b^3}\left[a + bx - 2a\log|a+bx| - \frac{a^2}{a+bx}\right] + C$

3. a. $\pm 4\left[\sin\frac{x}{4} - \cos\frac{x}{4}\right] + C$

b. $-2\left[\frac{(1-x)^{5/2}}{5} - \frac{2(1-x)^{3/2}}{3} + \sqrt{1-x}\right] + C$

4. $x\sin x + \cos x - \frac{1}{4}\cos 2x + C$

5. $\frac{e^x}{(x+1)^2} + C$

6. $-\left(1+\frac{1}{x^4}\right)^{1/4} + C$

7. $-2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$

8. $\frac{2}{\pi}[\sqrt{x-x^2} - (1-2x)\sin^{-1}\sqrt{x}] - x + C$

9. $-\log|y + \sqrt{y^2+1}| - \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2y^2+2} - y}{\sqrt{2y^2+2} + y}\right| + C$

where $\cot^2 x = 1 + y^2$

10. $\sqrt{2}\sin^{-1}(\sin x - \cos x) + C$

12. $\sin 2\theta \ln\sqrt{\dfrac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}} - \dfrac{1}{2}\ln|\sec 2\theta| + C$

13. $-\log\left(\dfrac{1 + xe^x}{xe^x}\right) - \dfrac{1}{1 + xe^x} + C$

14. $-4\left[\log\left|\dfrac{1 + \sqrt{1 - x}}{\sqrt{x}}\right| - \cos^{-1}\sqrt{x}\right] + C$

15. $\dfrac{1}{4}\log\left|\dfrac{x^2 + 1}{(x + 1)^2}\right| + \dfrac{3}{2}\tan^{-1}x + \dfrac{x}{1 + x^2} + C$

16. $\dfrac{3}{2}\left\{\dfrac{2}{3}(x + 1)\tan^{-1}\left(\dfrac{2}{3}(x + 1)\right) - \log\sqrt{4x^2 + 8x + 13}\right\} + C$

17. $\dfrac{1}{6(m+1)}\left(2x^{3m} + 3x^{2m} + 6x^m\right)^{(m+1)/m} + C$

Hints and Solutions

JEE Advanced

Single Correct Answer Type

1. c. Let $I = \int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx$

$$= \int \frac{(\cos^2 x + \cos^4 x) \cos x}{\sin^2 x (1 + \sin^2 x)} dx$$

$$= \int \frac{[1 - \sin^2 x + (1 - \sin^2 x)^2] \cos x}{\sin^2 x (1 + \sin^2 x)} dx$$

$$= \int \frac{(2 - 3\sin^2 x + \sin^4 x) \cos x}{\sin^2 x (1 + \sin^2 x)} dx$$

Put $\sin x = t$ or $\cos x \, dx = dt$

$$\therefore I = \int \frac{2 - 3t^2 + t^4}{t^4 + t^2} dt$$

$$= \int \left(1 + \frac{2}{t^2} - \frac{6}{t^2 + 1}\right) dt$$

$$= t - \frac{2}{t} - 6 \tan^{-1}(t) + C$$

$$= \sin x - 2 (\sin x)^{-1} - 6 \tan^{-1}(\sin x) + C$$

2. d. $I = \int \frac{x^2 - 1}{x^3 \sqrt{2x^4 - 2x^2 + 1}} dx$

$$= \frac{1}{4} \int \frac{\frac{4}{x^3} - \frac{4}{x^5}}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}} dx$$

Put $2 - \frac{2}{x^2} + \frac{1}{x^4} = t \quad \therefore \left(\frac{4}{x^3} - \frac{4}{x^5}\right) dx = dt$

$$\therefore I = \frac{1}{4} \int \frac{dt}{\sqrt{t}} = \frac{2\sqrt{t}}{4} + C$$

$$= \frac{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}}{2} + C$$

$$= \frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + C$$

3. a. Here $f(f(x)) = \frac{f(x)}{[1 + f(x)^n]^{1/n}}$

$$= \frac{x}{(1 + 2x^n)^{1/x}}$$

$$f(f(f(x))) = \frac{x}{(1 + 3x^n)^{1/n}}$$

$$\Rightarrow \quad g(x) = (\underbrace{fofo \ldots of}_{n \text{ times}})(x) = \frac{x}{(1 + nx^n)^{1/n}}$$

Hence, $I = \int x^{n-2}\, g(x)\,dx$

$$= \int \frac{x^{n-1}\,dx}{(1+nx^n)^{1/n}}$$

$$= \frac{1}{n^2}\int \frac{n^2 x^{n-1}\,dx}{(1+nx^n)^{1/n}}$$

$$= \frac{1}{n^2}\int \frac{\frac{d}{dx}(1+nx^n)}{(1+nx^n)^{1/n}}\,dx$$

$$\therefore \quad I = \frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}} + K$$

4. c. $J - I = \int \frac{e^x(e^{2x}-1)}{e^{4x}+e^{2x}+1}\,dx$

$$= \int \frac{(z^2-1)}{z^4+z^2+1}\,dz \text{ (where } z = e^x\text{)}$$

$$= \int \frac{\left(1-\frac{1}{z^2}\right)dz}{\left(z+\frac{1}{z}\right)^2 - 1}$$

$$= \int \frac{dt}{t^2-1} \quad \left(\text{where } z + \frac{1}{z} = t\right)$$

$$= \frac{1}{2}\log_e\left(\frac{t-1}{t+1}\right) + C$$

$$= \frac{1}{2}\log_e\left(\frac{e^x+e^{-x}-1}{e^x+e^{-x}+1}\right) + C$$

$$= \frac{1}{2}\log_e\left(\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right) + C$$

5. c. $I = \int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}}\,dx$

Let $\sec x + \tan x = t$

$\therefore \sec x - \tan x = 1/t$

Now, $(\sec x \tan x + \sec^2 x)\,dx = dt$

or $\sec x(\sec x + \tan x)\,dx = dt$

or $\sec x\,dx = \frac{dt}{t}, \frac{1}{2}\left(t + \frac{1}{t}\right) = \sec x$

$$\therefore I = \frac{1}{2}\int \frac{\left(t+\frac{1}{t}\right)}{t^{9/2}}\frac{dt}{t}$$

$$= \frac{1}{2}\int (t^{-9/2} + t^{-13/2})\,dt$$

$$= \frac{1}{2}\left[\frac{t^{-9/2+1}}{-\frac{9}{2}+1} + \frac{t^{-13/2+1}}{-\frac{13}{2}+1}\right] + K$$

$$= \frac{1}{2}\left[\frac{t^{-7/2}}{-\frac{7}{2}} + \frac{t^{-11/2}}{-\frac{11}{2}}\right] + K$$

$$= -\frac{1}{7}t^{-7/2} - \frac{1}{11}t^{-11/2} + K$$

$$= -\frac{1}{7}\frac{1}{t^{7/2}} - \frac{1}{11}\frac{1}{t^{11/2}} + K$$

$$= -\frac{1}{t^{11/2}}\left(\frac{1}{11} + \frac{t^2}{7}\right) = -\frac{1}{(\sec x + \tan x)^{11/2}}$$

$$\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$$

Assertion-Reasoning Type

1. d. $F(x) = \int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx$

$$\Rightarrow \quad F(x) = \frac{1}{4}(2x - \sin 2x) + c$$

Since $F(x+\pi) \neq F(x)$

Hence, statement 1 is false.

But statement 2 is true as $\sin^2 x$ is periodic with period π.

Fill in the Blanks Type

1. We have $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}\,dx = Ax + B\ln(9e^{2x} - 4) + C$

Differentiating both sides w.r.t. x, we get

$$\frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} = A + \frac{18\,B\,e^{2x}}{9e^{2x} - 4} = A + \frac{18\,B\,e^x}{9e^x - 4e^{-x}}$$

or $\frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} = \frac{(9A + 18B)e^x - 4Ae^{-x}}{9e^x - 4e^{-x}}$

or $9A + 18B = 4, -4A = 6$

or $A = -3/2$, $B = \left(4 + \frac{27}{2}\right)\frac{1}{18} = \frac{35}{36}$

C can have any real value.

Subjective Type

1. $I = \int \frac{\sin x}{\sin x - \cos x}\,dx$

$$= \frac{1}{2}\int \frac{2\sin x}{\sin x - \cos x}\,dx$$

$$= \frac{1}{2}\int \frac{\sin x + \cos x + \sin x - \cos x}{\sin x - \cos x}\,dx$$

$$= \int \frac{\cos x + \sin x}{\sin x - \cos x}\,dx + \frac{1}{2}\int dx$$

$$= \frac{1}{2}\log|\sin x - \cos x| + \frac{x}{2} + C$$

2. $I = \int \frac{x^2\,dx}{(a+bx)^2}$

Let $a + bx = t$ or $x = \left(\frac{t-a}{b}\right)$

or $dx = \frac{dt}{b}$

$$\therefore I = \frac{1}{b^3}\int \frac{t^2 - 2at + a^2}{t^2}\,dt$$

$$= \frac{1}{b^3}\int\left(1 - \frac{2a}{t} + \frac{a^2}{t^2}\right)dt$$

$$= \frac{1}{b^3}\left[t - 2a\log|t| - \frac{a^2}{t}\right] + C$$

$$= \frac{1}{b^3}\left[a + bx - 2a\log|a+bx| - \frac{a^2}{a+bx}\right] + C$$

3. a. $\int\sqrt{1+\sin\left(\frac{x}{2}\right)}dx = \int\sqrt{\sin^2\frac{x}{4} + \cos^2\frac{x}{4} + 2\sin\frac{x}{4}\cos\frac{x}{4}}\,dx$

$$= \pm\int\left(\sin\frac{x}{4} + \cos\frac{x}{4}\right)dx$$

$$= \pm\left[\frac{-\cos x/4}{1/4} + \frac{\sin x/4}{1/4}\right] + C$$

$$= \pm 4\left[\sin\frac{x}{4} - \cos\frac{x}{4}\right] + C$$

b. $I = \int\frac{x^2}{\sqrt{1-x}}\,dx$

Let $1 - x = t^2$ or $dx = -2t\,dt$

$$\therefore I = \int\frac{(1-t^2)^2}{t}(-2t)\,dt$$

$$= -2\int(t^4 - 2t^2 + 1)\,dt$$

$$= -2\left[\frac{t^5}{5} - \frac{2t^3}{3} + t\right] + C$$

$$= -2\left[\frac{(1-x)^{5/2}}{5} - \frac{2(1-x)^{3/2}}{3} + \sqrt{1-x}\right] + C$$

4. $\int(e^{\log x} + \sin x)\cos x\,dx$

$= \int(x + \sin x)\cos x\,dx$ [Using $e^{\log x} = x$]

$$= \int x\cos x + \frac{1}{2}\int\sin 2x\,dx$$

$$= x\sin x - \int\sin x\,dx + \frac{1}{2}\left(\frac{-\cos 2x}{2}\right) + C$$

$$= x\sin x + \cos x - \frac{1}{4}\cos 2x + C$$

Alternative Method:

$\int(e^{\log x} + \sin x)\cos x\,dx$

$= \int(x + \sin x)\cos x\,dx$

$= \int(x + \sin x)(1 + \cos x - 1)\,dx$

$= \int(x + \sin x)(1 + \cos x)\,dx - \int(x + \sin x)\,dx$

$$= \frac{(x+\sin x)^2}{2} - \frac{x^2}{2} + \cos x + C$$

$$= \frac{x^2 + 2x\sin x + \sin^2 x}{2} - \frac{x^2}{2} + \cos x + C$$

$$= \frac{2x\sin x + \sin^2 x}{2} + \cos x + C$$

$$= x\sin x + \frac{1 - \cos 2x}{4} + \cos x + C$$

$$= x\sin x - \frac{\cos 2x}{4} + \cos x + C$$

5. $I = \int\frac{(x-1)e^x}{(x+1)^3}\,dx$

$$= \int\frac{(x+1-2)e^x}{(x+1)^3}\,dx$$

$$= \int\left[\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}\right]e^x\,dx$$

$$= \int\left[\frac{1}{(x+1)^2} + \left(\frac{1}{(x+1)^2}\right)'\right]e^x\,dx$$

$$= \frac{e^x}{(x+1)^2} + C$$

6. Let $I = \frac{dx}{x^2(x^4+1)^{3/4}} = \int\frac{dx}{x^3\cdot x^2\left(1 + \frac{1}{x^4}\right)^{3/4}}$

Put $1 + \frac{1}{x^4} = t$ or $\frac{-4}{x^5}dx = dt$ or $\frac{dx}{x^5} = -\frac{dt}{4}$

$$\therefore I = \int\frac{-dt}{4t^{3/4}} = \frac{-1}{4}\frac{t^{-3/4+1}}{\frac{-3}{4}+1} + C$$

$$= -t^{1/4} + C = -\left(1 + \frac{1}{x^4}\right)^{1/4} + C$$

7. $I = \int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

Put $x = \cos^2\theta$ or $dx = -2\cos\theta\sin\theta\,d\theta$

$$\therefore I = -\int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,2\sin\theta\cos\theta\,d\theta$$

$$= -\int\frac{\sin\theta/2}{\cos\theta/2}\,2\cdot 2\sin\theta/2\cos\theta/2\cos\theta\,d\theta$$

$$= -2\int(1 - \cos\theta)\cos\theta\,d\theta$$

$$= -2\int(\cos\theta - \cos^2\theta)\,d\theta$$

$$= -2\int\left(\cos\theta - \frac{1 + \cos 2\theta}{2}\right)d\theta$$

$$= -2\left[\sin\theta - \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right] + C$$

$$= -2\sqrt{1-x} + \frac{2}{2}\left[\cos^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x}\right] + C$$

[Using $\sin\theta = \sqrt{1-x}$]

$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$$

8. Let $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\,dx$

We know that

$\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \pi/2$ (1)

Also, $\cos^{-1}\sqrt{x} = \pi/2 - \sin^{-1}\sqrt{x}$ (2)

Using equations (1) and (2), we get

$$I = \int \frac{\sin^{-1}\sqrt{x} - \left(\pi/2 - \sin^{-1}\sqrt{x}\right)}{\pi/2}\,dx$$

$$= \frac{2}{\pi}\int (2\sin^{-1}\sqrt{x} - \pi/2)\,dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx - \int 1\,dx$$

Let $x = \sin^2\theta$ or $dx = 2\sin\theta\cos\theta\,d\theta$

$$\therefore \quad I = \frac{4}{\pi}\int \sin^{-1}(\sin\theta)\,2\sin\theta\cos\theta\,d\theta - x + C$$

$$= \frac{4}{\pi}\int \theta\sin 2\theta\,d\theta - x + C$$

$$= \frac{4}{\pi}\left[\frac{-\theta\cos 2\theta}{2} + \int 1\times\frac{\cos 2\theta}{2}\,d\theta\right] - x + C$$

[Integrating by parts]

$$= \frac{4}{\pi}\left[\frac{-\theta\cos 2\theta}{2} + \frac{\sin 2\theta}{4}\right] - x + C$$

$$= \frac{4}{4\times\pi}[-2\sin^{-1}\sqrt{x}\,(1-2x) + 2\sqrt{x}\sqrt{1-x}] - x + C$$

$$= \frac{2}{\pi}[\sqrt{x-x^2} - (1-2x)\sin^{-1}\sqrt{x}] - x + C$$

9. $I = \int \frac{\sqrt{\cos 2x}}{\sin x}\,dx$

$$= \int \frac{\sqrt{\cos^2 x - \sin^2 x}}{\sin x}\,dx$$

$$= \int \sqrt{\cot^2 x - 1}\,dx$$

Put $\cot^2 x - 1 = y^2$

or $\cot^2 x = 1 + y^2$

or $-2\cot x\,\mathrm{cosec}^2 x\,dx = 2y\,dy$

or $dx = \dfrac{-y\,dy}{\sqrt{1+y^2}\,(2+y^2)}$

$$\therefore I = -\int \frac{y\times y\,dy}{\sqrt{1+y^2}\,(2+y^2)}$$

$$= -\int \frac{1}{\sqrt{y^2+1}}\,dy + 2\int \frac{dy}{(y^2+2)\sqrt{y^2+1}}$$

$$= -\log|y + \sqrt{y^2+1}| + 2I_1 \quad (1)$$

where $I_1 = \int \dfrac{dy}{(y^2+2)\sqrt{y^2+1}}$

Put $y = \dfrac{1}{t}$ or $dy = -\dfrac{dt}{t^2}$

$$\therefore I_1 = \int \frac{-\frac{dt}{t^2}}{\left(\frac{1}{t^2}+2\right)\sqrt{\frac{1}{t^2}+1}}$$

$$= -\int \frac{t\,dt}{\left(1+2t^2\right)\sqrt{t^2+1}}$$

Now let $t^2 + 1 = z^2$

or $t\,dt = z\,dz$

$$\therefore I_1 = -\int \frac{z\,dz}{\left(1+2(z^2-1)\right)z}$$

$$= -\int \frac{dz}{2z^2-1}$$

$$= -\frac{1}{2}\int \frac{dz}{z^2 - \frac{1}{2}}$$

$$= -\frac{1}{2\sqrt{2}}\log\left|\frac{z - \frac{1}{\sqrt{2}}}{z + \frac{1}{\sqrt{2}}}\right|$$

$$= -\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{\frac{1}{y^2}+1} - \frac{1}{\sqrt{2}}}{\sqrt{\frac{1}{y^2}+1} + \frac{1}{\sqrt{2}}}\right| + C$$

$$= -\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2y^2+2} - y}{\sqrt{2y^2+2} + y}\right| + C$$

$$\therefore \quad I = -\log|y + \sqrt{y^2+1}| - \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2y^2+2} - y}{\sqrt{2y^2+2} + y}\right| + C,$$

where $\cot^2 x = 1 + y^2$

10. $\int (\sqrt{\tan x} + \sqrt{\cot x})\,dx$

$$= \int \frac{\sin x + \cos x}{\sqrt{\sin x\cos x}}\,dx$$

$$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx$$

$$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}dx$$

$$= \sqrt{2}\int \frac{dt}{\sqrt{1-t^2}}, \text{ where } t = \sin x - \cos x$$

$$= \sqrt{2}\sin^{-1} t + C$$

$$= \sqrt{2}\sin^{-1}(\sin x - \cos x) + C$$

11. $\int\left(\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}} + \frac{\ln\left(1+\sqrt[6]{x}\right)}{\sqrt[3]{x}+\sqrt{x}}\right)dx$

$$\text{Let } I = \underbrace{\int \frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}dx}_{I_1} + \underbrace{\int \frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}dx}_{I_2} \quad (1)$$

$$I_1 = \int \frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}dx$$

Let $x = y^{12}$ so that $dx = 12\, y^{11}\, dy$

$$\therefore I_1 = \int \frac{12\, y^{11}}{y^4 + y^3}dy = 12\int \frac{y^8}{1+y}dy$$

$$= 12\int\left(y^7 - y^6 + y^5 - y^4 + y^3 - y^2 + y - 1 + \frac{1}{y+1}\right)dy$$

$$= 12\left[\frac{y^8}{8} - \frac{y^7}{7} + \frac{y^6}{6} - \frac{y^5}{5} + \frac{y^4}{4} - \frac{y^3}{3} + \frac{y^2}{2} - y + \log|y+1|\right] + C$$

$$= \frac{2}{3}x^{2/3} - \frac{12}{7}x^{7/12} + 2x^{1/2} - \frac{12}{5}x^{5/12} + 3x^{1/3}$$
$$- 4x^{1/4} + 6x^{1/6} - 12x^{1/12} + 12\log|x^{1/12}+1| + C_1 \quad (2)$$

$$I_2 = \int \frac{\ln(1+x^{1/6})}{x^{1/3}+x^{1/2}}dx$$

Let $x = z^6$ so that $dx = 6z^5\, dz$

$$\therefore I_2 = \int \frac{\ln(1+z)}{z^2+z^3}6z^5\, dz$$

$$= \int \frac{6z^3 \ln(z+1)}{z+1}dz$$

Put $z + 1 = t$ or $dz = dt$

$$\therefore I_2 = \int \frac{6(t-1)^3 \ln t}{t}dt$$

$$= 6\int\left(t^2 - 3t + 3 - \frac{1}{t}\right)\ln t\, dt$$

$$= 6\left[\int (t^2 - 3t + 3)\ln t\, dt - \int \frac{1}{t}\ln t\, dt\right]$$

$$= 6\left[\left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t\right)\ln t - \int\left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t\right)\frac{1}{t}dt - \frac{(\ln t)^2}{2}\right] + C$$

$$= 6\left[\left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t\right)\ln t - \left(\frac{t^3}{9} - \frac{3t^2}{4} + 3t\right) - \frac{(\ln t)^2}{2}\right] + C \quad (3)$$

Thus, we get the value of I on substituting the values of I_1 and I_2 from equations (2) and (3) in equation (1).

12. Let $\int \cos 2\theta \ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)d\theta$

$$= \ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)\int \cos 2\theta\, d\theta$$
$$- \int \frac{(\sin 2\theta)(\cos\theta - \sin\theta)}{2(\sin\theta + \cos\theta)}\frac{2}{(\cos\theta - \sin\theta)^2}$$

$$= \frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) - I_1$$

$$I_1 = \int \frac{(\sin 2\theta)}{(\sin\theta + \cos\theta)(-\sin\theta + \cos\theta)}d\theta$$

$$= \int \frac{\sin 2\theta}{\cos 2\theta}d\theta = \frac{1}{2}\ln|\sec 2\theta|$$

$$\therefore I = \sin 2\theta \ln\sqrt{\frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}} - \frac{1}{2}\ln|\sec 2\theta| + C$$

13. $I = \int \frac{(x+1)}{x(1+xe^x)^2}dx$

$$= \int \frac{e^x (x+1)}{x e^x (1+xe^x)^2}dx$$

Put $1 + xe^x = t$ or $(xe^x + e^x)\, dx = dt$

$$= \int \frac{dt}{(t-1)t^2}$$

$$= \int\left(\frac{1}{1-t} + \frac{1}{t} + \frac{1}{t^2}\right)dt \quad \text{(using partial fractions)}$$

$$= -\log|1-t| + \log|t| - \frac{1}{t} + C$$

$$= -\log\left|\frac{t}{1-t}\right| - \frac{1}{t} + C$$

$$= -\log\left|\frac{1+xe^x}{-xe^x}\right| - \frac{1}{1+xe^x} + C$$

$$= -\log\left(\frac{1+xe^x}{xe^x}\right) - \frac{1}{1+xe^x} + C$$

14. Put $x = \cos^2\theta$ or $dx = -2\cos\theta\sin\theta\, d\theta$

$$\therefore \int \frac{1}{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx = \int\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{1/2}\left(\frac{-2\cos\theta\sin\theta\, d\theta}{\cos^2\theta}\right)dx$$

$$= -\int \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\cdot\frac{4\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta}d\theta$$

$$= -\int \frac{4\sin^2\frac{\theta}{2}}{\cos\theta}d\theta$$

$$= -4\int \frac{1-\cos\theta}{\cos\theta}\,d\theta$$

$$= -4\int (\sec\theta - 1)\,d\theta$$

$$= -4\,[\log|\sec\theta + \tan\theta| - \theta] + C$$

$$= -4\left[\log\left|\frac{1}{\sqrt{x}} + \frac{\sqrt{1-x}}{\sqrt{x}}\right| - \cos^{-1}\sqrt{x}\right] + C$$

$$= -4\left[\log\left|\frac{1+\sqrt{1-x}}{\sqrt{x}}\right| - \cos^{-1}\sqrt{x}\right] + C$$

15. $\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}dx$

$$= \int \frac{x(x^2+1)+2(x+1)}{(x^2+1)^2(x+1)}\,dx$$

$$= \int \frac{x}{(x^2+1)(x+1)}\,dx + 2\int \frac{dx}{(x^2+1)^2}$$

$$= \int\left(\frac{x+1}{2(x^2+1)} - \frac{1}{2(x+1)}\right)dx + \int \frac{2}{(x^2+1)^2}\,dx$$

$$= \frac{1}{4}\int \frac{2x}{1+x^2}\,dx + \frac{1}{2}\int \frac{dx}{x^2+1} - \int \frac{dx}{2(x+1)} + \int \frac{2}{(x^2+1)^2}\,dx$$

$$= \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x - \frac{1}{2}\log|x+1| + 2I$$

where $I = \int \frac{dx}{(x^2+1)^2}$. Put $x = \tan\theta$ or $dx = \sec^2\theta\,d\theta$. Then

$$I = \int \frac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)^2}$$

$$= \int \cos^2\theta\,d\theta$$

$$= \int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{1}{2}\left(\theta + \frac{\tan\theta}{1+\tan^2\theta}\right) + C$$

$$= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{1+x^2} + C$$

$\therefore$ Given integral

$$= \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x - \frac{1}{2}\log|x+1| + \tan^{-1}x + \frac{x}{1+x^2} + C$$

$$= \frac{1}{4}\log\left|\frac{x^2+1}{(x+1)^2}\right| + \frac{3}{2}\tan^{-1}x + \frac{x}{1+x^2} + C$$

16. $I = \int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$

$$= \int \sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+3^2}}\right)dx$$

[Put $2x+2 = 3\tan\theta$ or $2\,dx = 3\sec^2\theta\,d\theta$]

$$= \int \sin^{-1}\left(\frac{3\tan\theta}{3\sec\theta}\right)\frac{3}{2}\sec^2\theta\,d\theta$$

$$= \frac{3}{2}\int \theta\sec^2\theta\,d\theta$$

$$= \frac{3}{2}\left\{\theta\tan\theta - \int \tan\theta\,d\theta\right\}$$

$$= \frac{3}{2}\{\theta\tan\theta - \log|\sec\theta|\} + C$$

$$= \frac{3}{2}\left\{\frac{2x+2}{3}\tan^{-1}\left(\frac{2x+2}{3}\right) - \log\left(\sqrt{1+\left(\frac{2x+2}{3}\right)^2}\right)\right\} + C$$

$$= \frac{3}{2}\left\{\frac{2}{3}(x+1)\tan^{-1}\left(\frac{2}{3}(x+1)\right) - \log\sqrt{4x^2+8x+13}\right\} + C$$

17. $I = \int \left(x^{3m}+x^{2m}+x^m\right)\left(2x^{2m}+3x^m+6\right)^{1/m}dx$

$$= \int \left(\frac{x^{3m}+x^{2m}+x^m}{x}\right)(x)\left(2x^{2m}+3x^m+6\right)^{1/m}dxdx$$

$$= \int \left(x^{3m-1}+x^{2m-1}+x^{m-1}\right)\left(2x^{3m}+3x^{2m}+6x^m\right)^{1/m}dx$$

Let $2x^{3m}+3x^{2m}+6x^m = t$ or $dt = 6m(x^{3m-1}+x^{2m-1}+x^{m-1})dx$

$$\therefore I = \int t^{1/m}\frac{dt}{6m} = \frac{1}{6m}\frac{t^{\frac{1}{m}+1}}{\frac{1}{m}+1} + C$$

$$= \frac{1}{6(m+1)}\left(2x^{3m}+3x^{2m}+6x^m\right)^{(m+1)/m} + C$$